Weighted Composition Operators on the Bloch Space

The Marriage of Two Operators

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Generalization of Two Operators

Two very important operators studied on Banach spaces of analytic functions are:

1. The *composition operator with symbol* φ

$$C_\varphi f = f \circ \varphi.$$

2. The *multiplication operator with symbol* ψ

$$M_\psi f = \psi \cdot f.$$

We can generalize these two operators by defining the *weighted composition operator* as

$$W_{\psi,\varphi} f = \psi C_\varphi f = \psi \cdot (f \circ \varphi).$$

Weighted Composition Operators in Mathematics

1. Evolution of the field of Composition Operators.

2. All isometries on H^p $(p \neq 2)$ are weighted composition operators where H^p is the set of all analytic functions on the disk such that

$$\sup_{0<r<1} \int_0^{2\pi} \left| f(re^{i\theta}) \right|^p \frac{d\theta}{2\pi} < \infty.$$

   F. Forelli, *The Isometries of H^p*, Canadian Journal of Math, 1964.

3. Weighted Composition Operators are tied to the classification of Dichotomies in Dynamical Systems.

   C. Chicone & Y. Latushkin, *Evolution Semigroups in Dynamical Systems and Differential Equations*, AMS Press, 1999.

Outline

1. Formalization of Weighted Composition Operators
2. Trends in the Study of Weighted Composition Operators
3. Boundedness of Weighted Composition Operators on the Bloch Space
4. Research Goals Concerning Weighted Composition Operators

Formalization of Weighted Composition Operators

Let $\Omega \subset \mathbb{C}^n$ be a domain (open and connected region) and let X be a Banach space of analytic functions on Ω.

Let φ be an analytic map from $\Omega \to \Omega$ and ψ be any analytic map on Ω. Then on the level of functions, both $f \circ \varphi$ and $\psi \cdot f$ make sense for any $f \in X$.

Fix $\varphi : \Omega \to \Omega$ analytic and ψ analytic on Ω. For $z \in \Omega$ and $f \in X$, define the *weighted compositon operator* $\psi C_\varphi : X \to Y$ by

$$(\psi C_\varphi f)(z) = \psi(z) \cdot (f(\varphi(z)))$$

where Y is some Banach space of analytic functions on Ω.

Trends in Studying Weighted Composition Operators

Driving Goal

The goal in studying any operator with symbol is to relate the function-theoretic properties of the symbol to the operator-theoretic properties of the operator.

For what ψ and φ is ψC_φ: bounded? invertible? isometric?

Other important concepts about ψC_φ:

1. estimates on norm
$$||\psi C_\varphi|| = \sup_{||f||=1} ||\psi C_\varphi f|| .$$
2. estimates on essential norm
$$||\psi C_\varphi||_e = \inf_{K \text{ compact}} ||\psi C_\varphi - K|| .$$
3. spectrum of ψC_φ.

Boundedness of ψC_φ on Spaces

We now consider the most fundamental concept to study about any operator... *boundedness*.

Definition

A linear operator $T : X \to Y$ between Banach spaces is bounded if there exists $M > 0$ such that

$$||Tf||_Y \leq M\,||f||_X .$$

What properties must ψ and φ possess for ψC_φ to be a bounded operator from X to X? The answer is dependent on two things:

1. Ω: the domain of $\mathbb{C}^n$.
2. X: the space of analytic functions on which ψC_φ is acting.

To frame our discussion, we will fix $\Omega = \mathbb{D}$ and consider the question of boundedness on:

1. Bloch space $\mathcal{B}$
2. little Bloch space $\mathcal{B}_0$

Weighted Composition Operators on the Bloch Space

The Bloch Space

A function f analytic in $\mathbb{D}$ is said to be *Bloch* if

$$\beta_f := \sup_{z\in\mathbb{D}} (1-|z|^2)\left|f'(z)\right| < \infty.$$

The space of Bloch functions, called the *Bloch space* $\mathcal{B}(\mathbb{D}) = \mathcal{B}$, is a Banach space under the norm $||f||_{\mathcal{B}} = |f(0)| + \beta_f$.

Examples

1. Polynomials.
2. H^∞, the set of bounded analytic functions on $\mathbb{D}$.
3. Analytic functions on $\mathbb{D}$ whose image has finite area.
4. $\log \dfrac{1+z}{1-z}$.

Why is this Hard?

Question

If we know the ψ for which M_ψ is bounded and the φ for which C_φ is bounded, won't these be all the symbols that make ψC_φ bounded?

Answer: Unfortunately No!

1. If M_ψ and C_φ are bounded, then ψC_φ is bounded.
2. This is not the only situation for which ψC_φ is bounded. Consider

$$\psi(z) = \log \frac{2}{1-z}$$
$$\varphi(z) = \frac{1-z}{2}.$$

We will see that M_ψ is **not bounded** on $\mathcal{B}$, C_φ is **bounded** on $\mathcal{B}$, but ψC_φ is **bounded** on $\mathcal{B}$.

Bounding the Norm of ψC_φ

Where to begin...

$$\begin{aligned} ||\psi C_\varphi|| &= \sup_{||f||_{\mathcal{B}}=1} ||\psi C_\varphi f||_{\mathcal{B}} \\ &= \sup_{||f||_{\mathcal{B}}=1} \left(|(\psi C_\varphi f)(0)| + \beta_{\psi C_\varphi f}\right) \\ &\leq \sup_{||f||_{\mathcal{B}}=1} |(\psi C_\varphi f)(0)| \\ &\quad + \sup_{||f||_{\mathcal{B}}=1} \left(\sup_{z\in\mathbb{D}}(1-|z|^2)\left|(\psi(z)f(\varphi(z)))'\right|\right) \end{aligned}$$

Goal

We want to determine what properties of ψ and φ make ψC_φ a bounded operator on $\mathcal{B}$. To do this, we will first look at how one might determine what properties of ψ and φ make the semi-norm bounded.

Let $f \in \mathcal{B}$ and $z \in \mathbb{D}$. Consider the quantity

$$\begin{aligned}(1-|z|^2)\left|(\psi C_\varphi f)'(z)\right| &= (1-|z|^2)\left|(\psi(z)f(\varphi(z)))'\right| \\ &= (1-|z|^2)\left|\psi'(z)f(\varphi(z)) + \psi(z)f'(\varphi(z))\varphi'(z)\right| \\ &\le (1-|z|^2)\left|\psi'(z)\right|\left|f(\varphi(z))\right| \\ &\quad + (1-|z|^2)\left|\psi(z)\varphi'(z)\right|\left|f'(\varphi(z))\right|.\end{aligned}$$

Observations

1. The estimate needs to be independent of f and dependent on $||f||_{\mathcal{B}}$.
2. If we can bound both parts individually, then the entire quantity will be bounded.

Useful Facts

Let $f \in \mathcal{B}$, φ analytic in $\mathbb{D}$ and $z \in \mathbb{D}$.

1. $|f(z)| \leq \dfrac{1}{\log 2} \log \dfrac{2}{1-|z|^2} \, ||f||_{\mathcal{B}}$.
2. $\beta_{f\circ\varphi} \leq \beta_f$.
3. $\beta_f \leq ||f||_{\mathcal{B}}$.

$$(1-|z|^2)\left|\psi'(z)\right| |f(\varphi(z))| \leq \frac{1}{\log 2}(1-|z|^2)\left|\psi'(z)\right| \log \frac{2}{1-|\varphi(z)|^2} \, ||f||_{\mathcal{B}}$$

$$\begin{aligned}
&(1-|z|^2)\,|f'(\varphi(z))|\,|\psi(z)\varphi'(z)| \\
&\qquad = \frac{1-|z|^2}{1-|\varphi(z)|^2}\left|\psi(z)\varphi'(z)\right|(1-|\varphi(z)|^2)\left|f'(\varphi(z))\right| \\
&\qquad \leq \frac{1-|z|^2}{1-|\varphi(z)|^2}\left|\psi(z)\varphi'(z)\right| ||f||_{\mathcal{B}}.
\end{aligned}$$

The Main Result

Theorem (Ohno & Zhao, 2001)

Let ψ be an analytic function on the unit disk $\mathbb{D}$ and φ an analytic self-map of $\mathbb{D}$. Then ψC_φ is bounded on the Bloch space $\mathcal{B}$ if and only if the following are satisfied:

1. $$\sup_{z\in\mathbb{D}}(1-|z|^2)\left|\psi'(z)\right|\log\frac{2}{1-|\varphi(z)|^2}<\infty,$$
2. $$\sup_{z\in\mathbb{D}}\frac{1-|z|^2}{1-|\varphi(z)|^2}\left|\psi(z)\varphi'(z)\right|<\infty.$$

What Do We Know About M_ψ

Boundedness of M_ψ

When $\varphi(z) = z$, then $\psi C_\varphi = M_\psi$, and we have M_ψ is bounded on $\mathcal{B}$ if and only if the following conditions are satisfied:

1. $\sup_{z\in\mathbb{D}}(1-|z|^2)\left|\psi'(z)\right|\log\dfrac{1}{1-|z|^2} < \infty$,
2. $\sup_{z\in\mathbb{D}}|\psi(z)| < \infty$.

This matches up with already known results.

Theorem ((Version 1) Brown & Shields, 1991)

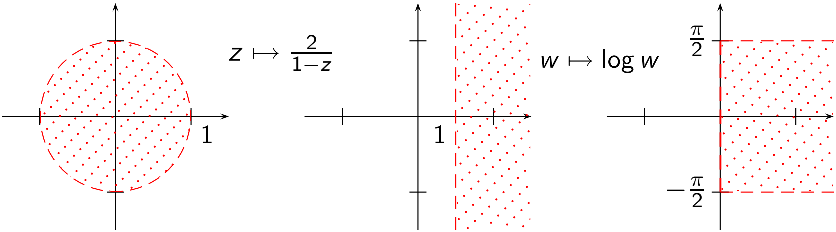
M_ψ is bounded on $\mathcal{B}$ if and only if $\psi \in H^\infty$ and

$$\left|\psi'(z)\right| = O\left(\frac{1}{(1-|z|)\log\frac{1}{1-|z|}}\right).$$

Back To The Hard Question

Recall

We said earlier that for $\psi(z) = \log \dfrac{2}{1-z}$, M_ψ is not bounded. Why?



Answer

Because ψ is not bounded.

Weighted Composition Operators on the Little Bloch Space

The little Bloch space $\mathcal{B}_0$

The *little Bloch space*

$$\mathcal{B}_0 = \left\{ f \in \mathcal{B} : \lim_{|z|\to 1^-} (1 - |z|^2) \left| f'(z) \right| = 0 \right\}$$

is a closed subspace of the Bloch space, and thus is a Banach space under the norm

$$\|f\|_{\mathcal{B}_0} = \|f\|_{\mathcal{B}}.$$

Examples of Little Bloch Functions

1. Polynomials: $f(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$.
2. Dilations: $g_r(z) = g(rz)$ for $g \in \mathcal{B}$ and $0 < r < 1$.

Properties of the Little Bloch Space

Proposition (Zhu, 2007)

Suppose $f \in \mathcal{B}$. Then $f \in \mathcal{B}_0$ iff $||f_r - f||_{\mathcal{B}} \to 0$ as $r \to 1^-$.

Corollary (Zhu, 2007)

$\mathcal{B}_0$ is the closure in $\mathcal{B}$ of the set of polynomials.

Sketch Proof.

1. f_r can be approximated by polynomials in H^∞.
2. $||\cdot||_{\mathcal{B}} \leq 2\,||\cdot||_{H^\infty}$.
3. So f_r can be approximated by polynomials in $\mathcal{B}$.

Proposition (Rubel & Shields, 1970)

$(\mathcal{B}_0)^{**} \cong \mathcal{B}$.

Goal

We wish to find the conditions on ψ and φ for which ψC_φ is a bounded operator on $\mathcal{B}_0$.

Theorem (Ohno & Zhao, 2001)

Let ψ be an analytic function on the unit disk $\mathbb{D}$ and φ and analytic self-map of $\mathbb{D}$. Then ψC_φ is bounded on the little Bloch space $\mathcal{B}_0$ if and only if the follow are all satisfied:

1. $\displaystyle\sup_{z\in\mathbb{D}}(1-|z|^2)\left|\psi'(z)\right|\log\frac{2}{1-|\varphi(z)|^2}<\infty,$
2. $\displaystyle\sup_{z\in\mathbb{D}}\frac{1-|z|^2}{1-|\varphi(z)|^2}\left|\psi(z)\varphi'(z)\right|<\infty.$
3. $\psi\in\mathcal{B}_0$
4. $\displaystyle\lim_{|z|\to 1^-}(1-|z|^2)\left|\psi(z)\varphi'(z)\right|=0.$

Definition

A complex-valued function ψ in $\mathbb{D}$ is called a multiplier on $\mathcal{B}(\mathcal{B}_0)$ if $\psi\mathcal{B} \subset \mathcal{B}$ $(\psi\mathcal{B}_0 \subset \mathcal{B}_0)$.

Theorem ((Full Version) Brown & Shields, 1991)

The following are equivalent:

1. ψ *is a multiplier on* $\mathcal{B}$
2. ψ *is a multiplier on* $\mathcal{B}_0$
3. $\psi \in H^\infty$ *and*

$$|\psi'(z)| = O\left(\frac{1}{(1-|z|)\log\frac{1}{1-|z|}}\right).$$

Research Goals

Norm Estimates: $? \leq ||\psi C_\varphi|| \leq ?$.

1. In [Xiong, 2004] established sharp bounds for C_φ on $\mathcal{B}$:

$$\max\left\{1, \frac{1}{2}\log\frac{1+|\varphi(0)|}{1-|\varphi(0)|}\right\} \leq ||C_\varphi|| \leq \max\left\{1, \frac{1}{2}\log\frac{1+|\varphi(0)|}{1-|\varphi(0)|} + \tau_\varphi^\infty\right\}$$

where $\tau_\varphi^\infty = \sup_{z\in\mathbb{D}}\left\{\frac{1-|z|^2}{1-|\varphi(z)|^2}|\varphi'(z)|\right\}$.

2. In [A. & Colonna, 2007] established bounds on C_φ on $\mathcal{B}$ in higher dimensions:
   - Sharp bounds on $\mathcal{B}$ on the unit ball $\mathbb{B}_n$ which reduce to that above.
   - Bounds on $\mathcal{B}$ on the unit polydisk $\mathbb{D}^n$.

3. There are no norm estimates for M_ψ on $\mathcal{B}$ as of yet.

More Research Goals

Isometries: Conditions for which ψC_φ is an isometry

$$||\psi C_\varphi f||_{\mathcal{B}} = ||f||_{\mathcal{B}}.$$

1. The isometric composition operators on $\mathcal{B}(\mathbb{D})$ are classified.
   F. Colonna, *Characterization of the Isometric Composition Operators on the Bloch Space*, Bull. of Australian Math Soc, 2005.
2. In [A. & Colonna, 2006] conditions are given for C_φ to be isometry on $\mathcal{B}$ in higher dimensions.
3. No conditions for M_ψ to be isometric.

Spectrum: Spectrum and spectral radius of ψC_φ, C_φ and M_ψ.

1. Isometric Case:
   - Spectral Radius is 1.
   - spectrum is $\overline{\mathbb{D}}$ if surjective.
   - spectrum is subset of $\partial\mathbb{D}$ otherwise.
2. Non-Isometric Case.

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